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On The Approximation Properties of Engel And Pierce Continued Fraction Expansions

H.W.S.S.Jayathissa[1], A.A.I.Perera[2], K.M.N.M.Chaturanga[3]
Department Of Mathematics, Faculty Of Science, University Of Peradeniya, Sri Lanka

Abstract: This study discusses some approximation properties of both Engel and Pierce expansions of an irrational number by a sequence of rational and the convergence nature of those sequences. The proposed work, found many rational approximation properties of both Engel and Pierce expansions of a given irrational. Thus, two integerssequences (p_n) and (q_n) are found in both Engel and Pierce expansions satisfy some properties. Furthermore, ratio sequence of above two sequencescalled sequence of convergentindicatesa sequence of rational numbers that converge to the given irrational. Also, the error between the given irrational and terms of its sequence of convergent in Engel and Pierce expansions are discussed. Moreover, every even convergent of a Pierce continued fraction is greater than odd convergent. In contrast, every convergent in the Engel continued fraction is increasing. We conjecture that convergent rate of the Pierce continued fraction quickly than other two continued fractions.

Keywords: Continued fraction, Convergent, Engel expansion, rational approximation, Pierce expansion.

I. INTRODUCTION

A simple continued fraction of a real number $x \in (0,1]$ is an expression of the form $x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \Lambda}}$ or $x = [a_0, a_1, \Lambda]$, which is denoted by $\underset{n=0}{\overset{\infty}{K}}(a_n)^{-1}$, where (a_n) is a sequence of positive integers. Simple continued fraction of $x = \frac{1}{x_0}$,this can be uniquely determined by the eliminating integers x_1, x_2, Λ , generating equations $x_0 = a_0 + \frac{1}{x_1},\ x_1 = a_1 + \frac{1}{x_2}, \Lambda$. [1]. The Engel expansion of x is of the form $x = \frac{1}{a_0} + \frac{1}{a_0 a_1} + \frac{1}{a_0 a_1 a_2} + \Lambda = \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n} a_k\right)^{-1}$, where (a_n) is a non-decreasing sequence of positive

integers and this can be uniquely determined by the eliminating integers x_1, x_2, Λ , generating equations $x = \frac{1}{a_0}(1 + x_1),\ x_1 = \frac{1}{a_1}(1 + x_2), \Lambda$. If this expansion is finite, it is called a finite Engle continued fraction [2] . Similarly, the Pierce expansion is given by $x = \frac{1}{a_0} - \frac{1}{a_0 a_1} + \frac{1}{a_0 a_1 a_2} - \Lambda = \sum_{n=0}^{\infty}(-1)^n\left(\prod_{k=0}^{n} a_k\right)^{-1}$ it

can be obtained by generating equations $x = \frac{1}{a_0}(1 - x_1),\ x_1 = \frac{1}{a_1}(1 - x_2), \Lambda$.[3]

Engel expansion of a real number $x \in (0,1]$ can be expressed as an ascending continued fraction with the form $x = \dfrac{1+\dfrac{1+\dfrac{1+\Lambda}{a_2}}{a_1}}{a_0}$ or $x = \{a_0, a_1, a_2, \Lambda\}$. The n^{th} convergent of the Engel continued fraction is defined by $c_n = \dfrac{1+\dfrac{1+\Lambda+\dfrac{1}{a_n}}{a_1}}{a_0}$ $\forall n = 0,1,2,\Lambda$, where (a_n) is a sequence of positive integers. [2]

Pierce expansion of a real number $x \in (0,1]$ can be expressed as an ascending alternating continued fraction with the form $x = \dfrac{1-\dfrac{1+\dfrac{1-\Lambda}{a_2}}{a_1}}{a_0}$ or $x = \langle a_0, a_1, a_2, \Lambda \rangle$.

The n^{th} convergent of the Pierce continued fraction is defined by $c_n = \dfrac{1-\dfrac{1+\Lambda+\dfrac{(-1)^n}{a_n}}{a_1}}{a_0}$ $\forall n = 0,1,2,\Lambda$, where (a_n) is a sequence of positive integers.

It is well known that the simple continued fraction is finite if and only if x is a rational and it is periodic if and only if x is a quadratic irrational. Simple infinite continued fraction expansions can be used to approximate irrationals. [1]

This study discusses the similar approximation properties of both Engel and Pierce expansions of a given irrational number in the interval $(0,1)$ and the convergence nature of sequence of rational numbers obtained using above mentioned expansions.

II. RESULTS AND DISCUSSION

2.1 Engel Continued Fraction

Proposition 1

If (p_n) and (q_n) are two sequences of positive integers given by $p_0 = 1,\ q_0 = a_0$ and $p_n = a_n p_{n-1} + 1,\ \ q_n = a_n q_{n-1}$, then $c_n = \dfrac{1}{a_0} + \dfrac{1}{a_0 a_1} + \Lambda + \dfrac{1}{a_0 a_1 a_2 \Lambda\ a_n} = \dfrac{p_n}{q_n}$ forall $n \in \mathrm{N}$.

Proof:

We prove this by mathematical induction. The result follows obviously for $n = 0$ and, when $n = 1$

$$c_1 = \{a_0, a_1\} = \frac{1+\dfrac{1}{a_1}}{a_0} = \frac{a_1+1}{a_0 a_1} = \frac{p_1}{q_1}$$

Therefore, the assertion is true for $n = 1$.

Assume that the assertion is true for $n = k$ some positive integer.

$$c_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + 1}{a_k q_{k-1}}$$

Consider

Now by $c_{k+1} = \{a_0, a_1, \Lambda\ , a_k, a_{k+1}\} = \left\{a_0, a_1, \Lambda\ , a_{k-1}, \dfrac{a_k a_{k+1}}{a_{k+1}+1}\right\}$ the induction hypothesis

$$c_{k+1} = \frac{\left(\frac{a_k a_{k+1}}{a_{k+1}+1}\right)p_{k-1}+1}{\left(\frac{a_k a_{k+1}}{a_{k+1}+1}\right)q_{k-1}} = \frac{a_k a_{k+1} p_{k-1} + a_{k+1} + 1}{a_k a_{k+1} q_{k-1}} = \frac{a_{k+1}\left(a_k p_{k-1}+1\right)+1}{a_{k+1}\left(a_k q_{k-1}\right)} = \frac{a_{k+1}p_k + 1}{a_{k+1}q_k} = \frac{p_{k+1}}{q_{k+1}}$$

Therefore,By the Principle of Mathematical Induction the assertion is true for all $n \in \mathrm{N}$.
The sequences (p_n) and (q_n) has following properties:

i. $p_n q_{n-1} - p_{n-1} q_n = q_{n-1}$ for all $n > 0.$
ii. $p_n q_{n-2} - p_{n-2} q_n = (a_n + 1) q_{n-2}$ for all $n > 1.$

One can prove the first identity using mathematical induction and second identity immediately follows by rearranging the terms of first identity. Based on these remarkablysimple recurrence properties, we can obtain one of our main results. Multiplying first identity by $\frac{1}{q_{n-1}q_n}$ and the second identity by $\frac{1}{q_{n-2}q_n}$ can obtain, $c_n = c_{n-1} + \frac{1}{q_n} = c_{n-2} + \frac{a_n+1}{q_n}$ for all $n > 1.$ Itshow that c_n s are form a strictly increasing sequence of rational numbers which is bounded aboveby x. So, monotone convergent theorem gives us the convergence of the sequence (c_n) and moreover we prove that limit of this sequence is $x.$

Proposition 2
If $x = \{a_0, a_1, a_2, \Lambda\ , a_n, \Lambda\ \}$, then $\lim_{n\to\infty} c_n = x.$
Proof:
Observe that c_{n+1} lies between c_n and x, therefore we have $\frac{1}{q_{n+1}} = c_{n+1} - c_n < x - c_n < x - c_{n-1}.$
Let $x = \{a_0, a_1, a_2, \Lambda\ , a_n, a_{n+1}, \Lambda\ \} = \{a_0, a_1, a_2, \Lambda\ , a_n, y\}$, where $y = \{a_{n+1}, a_{n+2}, a_2, \Lambda\ \Lambda\ \}$ as a real-valued constant. Then

$$x - c_n = \frac{yp_n + 1}{yq_n} - \frac{p_n}{q_n} = \frac{1}{yq_n} = \frac{1}{ya_0 a_1 \Lambda\ a_n}$$

Since (a_n) is a non-decreasing sequence of positive integers, by the squeeze theorem we have the desire answer.

2.2 Pierce Continued Fraction
Proposition 3
Let (p_n) and (q_n) be the two sequences of positive integers given by $p_0 = 1,\ q_0 = a_0$ and $p_n = a_n p_{n-1} + (-1)^n,\ \ q_n = a_n q_{n-1}$, then $c_n\ \frac{1}{a_0} - \frac{1}{a_0 a_1} + \Lambda\ + \frac{(-1)^n}{a_0 a_1 a_2 \Lambda\ a_n} = \frac{p_n}{q_n}$ for all $n \in \mathrm{N}$
Proof:

As in the previous case, here we argue by mathematical induction. The result follows obviously for $n = 0$ and, when $n = 1,$

$$c_1 = \langle a_0, a_1 \rangle = \frac{1 - \frac{1}{a_1}}{a_0} = \frac{a_1 - 1}{a_0 a_1} = \frac{p_1}{q_1}$$

Therefore, the assertion is true for $n = 1.$ Assume that the assertion is true for $n = k$, some positive integer.

$$c_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + (-1)^k}{a_k q_{k-1}}$$

According to the hypothesis,

$$c_{k+1} = \langle a_0, a_1, \Lambda\ , a_k, a_{k+1} \rangle = \left\langle a_0, a_1, \Lambda\ , a_{k-1}, \frac{a_k a_{k+1}}{a_{k+1} - 1} \right\rangle = \frac{\left(\frac{a_k a_{k+1}}{a_{k+1} - 1} \right) p_{k-1} + (-1)^k}{\left(\frac{a_k a_{k+1}}{a_{k+1} - 1} \right) q_{k-1}}$$

$$c_{k+1} = \frac{a_k a_{k+1} p_{k-1} + (-1)^k (a_{k+1} - 1)}{a_k a_{k+1} q_{k-1}} = \frac{a_{k+1}\left(a_k p_{k-1} + (-1)^k\right) + (-1)^{k+1}}{a_{k+1}(a_k q_{k-1})} = \frac{a_{k+1} p_k + (-1)^{k+1}}{a_{k+1} q_k} = \frac{p_{k+1}}{q_{k+1}}$$

Therefore, By the Principle of Mathematical Induction the assertion is true for all $n \in \mathrm{N}$.

The sequences (p_n) and (q_n) satisfy property of $p_n q_{n-1} - p_{n-1} q_n = (-1)^n q_{n-1}$ that can prove using mathematical inductionand also the sequences satisfy property of $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1}(a_n - 1) q_{n-2}$ it can be proven thatrearranging above property. Dividing first property by $q_n q_{n-1}$ and the second property by $q_n q_{n-2}$ we can obtain $c_n = c_{n-1} + \frac{(-1)^n}{q_n}$ and $c_n = c_{n-2} + \frac{(-1)^{n-1}(a_n - 1)}{q_n}$ respectively for all $n > 1.$ Moreover, we prove that every even convergent of a Pierce continued fraction is greater than odd convergent and limit of this sequence is $x.$

Proposition 4

Let $x = \langle a_0, a_1, a_2, \Lambda\ , a_n, \Lambda \rangle$ be an infinitePierce continued fraction, then $c_1 < c_3 < c_5 < c_7 < \Lambda\ \Lambda\ < x < \Lambda\ \Lambda\ < c_6 < c_4 < c_2 < c_0$, furthermore, $\lim_{n \to \infty} c_n = x.$

Proof:

Observe that $c_n - c_{n-2} = \frac{(-1)^{n-1}(a_n - 1)}{q_n} \forall n \geq 2.$

If n is odd, $c_{n-2} < c_n$ or $c_1 < c_3 < c_5 < \Lambda\ \Lambda\ (1)$

If n is even, $c_{n-2} > c_n$ or $c_0 > c_2 > c_4 > \Lambda\ \Lambda\ (2)$

Also, we know that $c_n - c_{n-1} = \frac{(-1)^n}{q_n} \forall n \geq 1$, this implies $c_{2k} > c_{2k-1} \Lambda\ \Lambda\ (3)$

Take two integers r and $s.$ If $r > s$, then by (1) and $(3),$ $c_{2r} > c_{2r-1} > c_{2s-1}.$ If $r < s$, then by (2) and $(3),$ $c_{2r} > c_{2s} > c_{2s-1}$ for any positive integer r and $s,$ we have $c_{2r} > c_{2s-1}$, this shows that $c_1 < c_3 < c_5 < \Lambda\ \Lambda\ < x < \Lambda\ \Lambda\ < c_4 < c_2 < c_0.$

Since (c_{2n-1}) is an increasing sequence bounded above by any even convergent,by the monotone convergent theorem $\lim_{n \to \infty} c_{2n-1}$ exists.That is $c_1 < c_3 < c_5 < \Lambda\ \Lambda\ < \lim_{n \to \infty} c_{2n-1}.$ similarly, (c_{2n}) is decreasing sequence bounded below by any odd convergent. Therefore $\lim_{n \to \infty} c_{2n}$ exists and $c_0 > c_2 > c_4 > \Lambda\ \Lambda\ > \lim_{n \to \infty} c_{2n}.$

Since $c_{2n} - c_{2n-1} = \frac{(-1)^{2n}}{q_{2n}}$ and $\lim_{n \to \infty} q_n = 0,$ we have $\lim_{n \to \infty} c_n = x.$

Moreover, one can estimate the error, as the previous case, by using the set of inequalities:
$$\frac{1}{q_{n+1}} = |c_{n+1} - c_n| > |x - c_n| > |x - c_{n-1}| \text{ and } |x - c_n| = \frac{1}{\langle a_{n+1}, a_{n+2}, \ldots \rangle a_0 a_1 \ldots a_n}.$$

To validate the above results, the variation of c_n for $\sqrt{23}$ is given below for the simple, Engle and Pierce continued fraction.

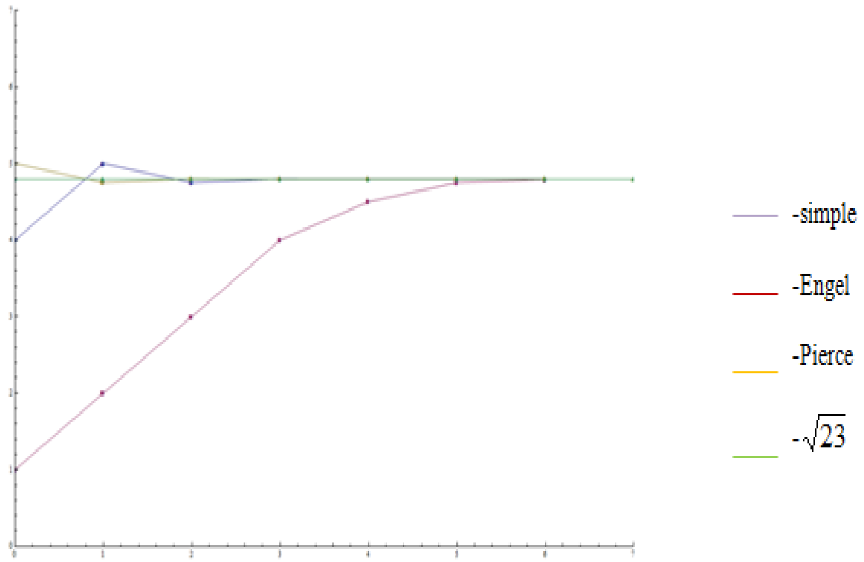

Figure 01 – The graph of the variations of c_n for different n

III. CONCLUSION

The proposed work indicated many approximation properties of both Engel and Pierce expansions of a given irrational. As a result, the sequences $p_n = a_n p_{n-1} + 1$ and $q_n = a_n q_{n-1}$ are found for all rational and irrationals in Engel expansion and they satisfy $p_n q_{n-1} - p_{n-1} q_n = q_{n-1}$, where $p_0 = 1$, $q_0 = a_0$ and $n \geq 0$. Whereas for the Pierce continued fractions, the above sequences can be seen as $p_n = a_n p_{n-1} + (-1)^n$ and $q_n = a_n q_{n-1}$ which satisfy the relationship of $p_n q_{n-1} - p_{n-1} q_n = (-1)^n q_{n-1}$, where $p_0 = 1$, $q_0 = a_0$ and $n \geq 0$. Furthermore, in Engel expansion, the difference between two consecutive convergent can be seen as $c_n - c_{n-1} = \frac{1}{q_n}$ and that for Pierce expansion as $c_n - c_{n-1} = \frac{(-1)^n}{q_n}$.

Also, the difference between the fraction and its convergent in Engel is found to be $\frac{1}{q_{n+1}} < x - c_n$ and $|x - c_n| < \frac{1}{q_{n+1}}$ in Pierce. Moreover, every even convergent of a Pierce continued fraction is greater than odd convergent. In addition, every convergent in the Engel continued fraction is increasing. Both Engel and Pierce continued fractions resulted $\lim c_n = x$, which is convergent. We hope to prove that convergent rate of the Pierce continued fraction quickly than other two continued fractions.

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